

The Constructivist
Fall 2005

Vol. 16, No. 1
ISSN 1091-4072

Constructivism Revisited: Implications and Reflections*

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The last century brought with it many scientific and technological advances. As we enter the 21st century, the models we currently use to describe and explain our world have dramatically changed: from Cartesian coordinate systems and Newtonian mechanics, to relativity; from quantum mechanics, to chaos, complexity, and string theories.

The fields of biology, neuroscience, cognitive psychology, and epistemology have been affected no less, and these changing models have, in turn, dramatically affected the way we now understand the interrelated acts of teaching and learning. In the early part of the twentieth century, “a change in behavior” was thought to define learning. Thus teaching was characterized as clear communication with appropriate learner practice, reinforcement, and motivation. Disciplines were broken down into skills and concepts, sequenced from what was considered simple to complex tasks, and assessments were designed to measure changes in behavior. We thought of the mind as a muscle in need of exercise and affected by practice. Today we know such models to be insufficient. Teaching and learning are far more complex; they are about interaction, growth, and development. Today we see “mind” as the result of the human construction of coherence, of explanation within communities of discourse as problems are posed and solved (Brown, 2001; Maturana and Varela, 1998; Deutsch, 2001).

When ACT was founded, the expressed purpose was to explicate and illustrate the educational implications of an emerging and controversial new theory in cognitive science: constructivism. Ten years later, the theory is no longer controversial. Most contemporary neurobiologists and cognitive scientists agree: knowledge is actively constructed. The implications of

constructivism for *education*, however, remain controversial. In the last ten years schools have been engaged in a flurry of reform initiatives encompassing new curricula, new forms of assessment, new standards, and professional development. But various interpretations of constructivism abound, often equating it with “hands-on” learning, discovery, and a host of pedagogical strategies. Resulting confusion and misinterpretation has resulted in public attacks by the media, by parents, and even at times various groups in the academic community. Thus it seems appropriate in this speech, to reflect on the biological and cognitive science evidence and to provide further implications for an application of the theory to education.

REVISITING THE THEORY

The Biological Foundation—Cells as Open Systems

What were the changes during the last century in biology? Current biologists no longer speak of simple genetic determinism, where new cells are produced from naked DNA. Instead they describe whole cellular networks—cellular structures as open systems—living, evolving systems that are organizationally closed but materially and energetically open (Capra, 2002). Cells need to feed on continual flows of matter and energy from their environment to stay alive. The Nobel laureate, Ilya Prigogine (1996), describes these cellular structures as dissipative, to emphasize the interplay between maintaining the organization of the structure of the cell on the one hand, and the flow and change (dissipation) on the other. When the flow of energy increases, the system may encounter a point of instability, known as a “bifurcation point,” at which point an entirely new form of order may emerge. Capra describes the importance of this new biological model:

This spontaneous emergence of order at critical points of instability is one of the most important concepts of the new understanding of life. It is technically known as self-organization and is often referred to simply as ‘emergence.’ It has been recognized as the dynamic origin of development, learning, and evolution. In other words, creativity—the generation of new forms—is a key property of all living systems. And since emergence is an integral part of the dynamics of open systems, we reach the important conclusion that open systems develop and evolve. Life constantly reaches

out into novelty. The theory of dissipative structures, formulated in terms of non-linear dynamics, explains not only the spontaneous emergence of order, but also helps us to define complexity. Whereas traditionally the study of complexity has been a study of complex structures, the focus is now shifting from the structures to the processes of their emergence. (2002, pg. 14)

Cells are now understood as being a part of an *autopoietic, non-linear, system*. An autopoietic system undergoes continual structural changes while preserving its web-like pattern of organization. The parts of this system continually produce and transform one another in two distinct ways: 1) by self renewal (tissues and organs replace cells in order to maintain a pattern of organization); and, 2) by creating new structures, which are developmental and a consequence of the coupling of the organism's cells with environmental influences. In other words, as living systems couple with their environments, recurrent interactions eventually result in structural changes. A cell membrane continually incorporates substances from its environment into its metabolic processes. An organism's nervous system changes its connectivity with every sense perception, but it also attempts to preserve coherence. This pattern is non-linear, and most importantly, the environment only triggers the structural changes; it does not produce them.

This current biological model of the emergence of novelty has become recognized as an explanation of the dynamic origin of development, learning, and evolution. During the last twenty-five years, the study of "mind" from this systemic, biological perspective blossomed into the rich, new interdisciplinary field of cognitive science—a field that integrates and transcends the traditional frameworks of child development, psychology, neurobiology, and epistemology.

The Cognitive Science Foundation

Early work in cognitive science focused on Cartesian views: dependent and independent variables, linear change models, and feedback loops. But as the field began to come into its own, and as research accumulated, these models were seen to be inadequate to explain the complexity of learning. More promise appeared in models characteristic of autopoietic systems. Researchers began to talk about the behavior of an organism as being

determined by its structure—structures that are themselves formed by a succession of structural changes (Piaget, 1987; Maturana and Varela, 1998).

Humans seek coherence and meaning. They act on and within their environments with strategies, or schemes, as they seek to make their world similar and maintain their organization (their understanding of it). When puzzled—when new problems emerge that contradict earlier notions, or when new problems make earlier strategies insufficient (or at a minimum inefficient), “bifurcations” result and new structures evolve. It is now commonly understood that human organisms act on their world, coupling with it, interpreting every experience. They do not simply take in, or absorb, information. They interpret it, organize it, and infer about it with the cognitive structures they have previously constructed. Thus, consciousness progressively evolves (Malerstein, 1986). It requires a level of cognitive abstraction that includes the progressive ability to hold and use mental images. These mental images allow us to formulate values, beliefs, goals, and strategies. And, representation of these ideas with language and mathematical symbols for explanation and justification within social communities of inquiry and discourse is what learning—epistemological evolution—is all about. This is a constructivist model of learning.

REVISITING THE IMPLICATIONS FOR EDUCATION

Constructivist Pedagogy vs. Constructivism as a Learning Theory

As constructivism began to take hold among cognitive scientists as a viable model to explain epistemology and behavior, it began to have an effect on models of education. A constructivist pedagogy began to be formulated and major reform began taking place. Classrooms soon became workshops, with teachers as facilitators, rather than transmitters of knowledge. The role of questioning, disequilibrium, learners paraphrasing each other and discussing ideas in learning communities, the importance of think time and pair talk, and the role of problem-solving and inquiry all began to be descriptive of the “new” classroom.

Although educators now commonly talk about a “constructivist-based” practice as if there is such a thing, in reality constructivism is not a theory of teaching; it is a theory about learning. In fact, as we shift our teaching

towards trying to support cognitive construction, the field of education has been left without well-articulated theories of teaching. Reformed practice has been attacked as fuzzy and relative. Major questions loom around *what* should be taught, *how* we should teach, and *how best to educate teachers* for this paradigmatic shift. The problem is that reform-based pedagogical strategies can be used without the desired learning necessarily resulting. This is because constructivism is a theory of learning, not a theory of teaching, and many educators who attempt to use such pedagogical strategies confuse discovery learning and “hands-on” approaches with constructivism.

What is constructivism? What implications does this theory have for teaching? How is it different from discovery learning? From a constructivist perspective, we cannot direct learning to get everyone to the same “ah ha” at the end of the lesson. We can only facilitate “coupling” with problematic situations, help raise questions and puzzlements, and support discourse and development. Learning—deep, conceptual learning—is about structural shifts in cognition. It is about self-organizing at moments of criticality. These changes are complex and non-linear, and they are the result of interacting, autopoietic systems.

From a constructivist perspective, meaning is understood to be the result of humans setting up relationships, reflecting on their actions, and modeling and constructing explanations. The reality of a cat is a different one than the reality of humans because cats’ and humans’ minds organize and infer differently. An infant has a different reality than an adult. The infant “knows” the world through sensorimotor schemes, in contrast to the adult who has highly developed mental imagery and logical structures of thought. Even amongst two adults realities are different, because they are based on interpretations that are the result of past individual experiences and beliefs which formed different neural networks and pathways. Within our human communities we test our models, construct explanations, and prove our thinking. In a sense, we negotiate meaning until we come to believe that we all mean the same thing. Thus constructivism is not to be confused with solipsism. However, as von Glasersfeld (1996) makes clear, we do not and cannot “share” meaning.

If two people share a room, there is one room and both live in it. If they share a bowl of cherries, none of the cherries is eaten by both persons. This is an important difference, and it must be

borne in mind when one speaks of shared meanings. The conceptual structures that constitute meanings or knowledge are not entities that could be used alternatively by different individuals. They are constructs that each user has to build up for him-or herself. And because they are individual constructs, one can never say whether or not two people have produced the same construct. (pg. 4).

The best we can do is discuss our ideas in communities until we believe that we agree, until the meanings seem “taken-as-shared” (Cobb, 1996), until our ideas are justified and accepted as explanations by the community. Since these individual constructs are built from reflection on strategies (schemes), on ideas (structures), and on models (representations), learning needs to be understood as the individual development of strategies, “big ideas”, and models, but within a cultural, social community of discourse.

The Dutch Version of Reform: Realistic Mathematics Education

While all of this reform work in education based on constructivism was going on in many parts of the world, the Dutch were involved in reform, too—most specifically in the field of mathematics education. But they were taking a slightly different tack. Hans Freudenthal, a Dutch mathematician, was invited by Bruner to participate in the Woods Hole conference on new math in 1960, but declined the offer. He argued against structuralism saying that mathematics should be thought of as a human activity of “mathematizing”—not as a discipline of structures to be transmitted, discovered, or even constructed—but as *schematizing, structuring, and modeling the world mathematically*. Thus, even as early as the sixties he took a constructivist view on education and began to work with dutch mathematics educators and researchers to develop what he called, Realistic Mathematics Education. Volumes of research and curricula were developed after extensive research, much too much to present coherently here. Suffice it to say that the didactics of RME, i.e. the use of context and models, intrigued me and caused me to seek funding for our now ongoing collaboration.

The Role of Context

In the United States, as we worked to reform mathematics education from a constructivist perspective, we were using good problems, at least we were attempting to! We knew rich exploration and conversation around big ideas were important to mathematical learning. But it is not enough to just come up with good problematic situations because these only bring forth *initial* schematizing. The teacher's responsibility is to foster growth over time *beyond* the student's initial attempts. The Dutch were employing context as a didactic to accomplish this. They were building in constraints and potentially realized suggestions to actively promote disequilibrium, reflection, and development. They were designing contexts that had the potential to perturb.

To take a simple illustration, the pictures in Figures 1, 2, and 3 show ways contexts might be used to foster progressive mathematizing around multiplication. When children are engaged in investigating "how many," the context of fruit trays [see Figure 1] allows for counting by ones, but the array arrangements support the construction of skip counting and/or using doubles. In contrast, the context of window shades supports the development of doubling and the use of the associative property [see Figure 2] simply because one curtain is pulled back; and the context of tiled-patios with furniture covering some of the tiles might support the development of the distributive property because some tiles are obscured and cannot be counted [see Figure 3]. By choosing and designing situations that have the potential to perturb learners' initial strategies, teachers can employ context as a powerful didactic that supports the development of big ideas (structuring) and strategies (schematizing). [Note: Similar pictures can be found in the Dutch curriculum, *Rekenen and Wiskunde* (van Galen et. al., 1985)].

The Role of Models

A second important aspect of the RME didactic is the use of models, such as the open number line, the ratio table, the double number line, and the bar model. Rather than using models as a visual or hands-on material as we had done in the United States in the early days of constructivism, within RME they are developed initially with contexts such as measurement (for the open number line), recipes and fair sharing (for the ratio table), and population bars and racetracks (for the bar model and double number line). After the models are developed, they are then employed by teachers as a community

tool to represent students' strategies for computation. Eventually these mathematical models become tools for learners to think with. Figure 4 shows an example of a child's work where the number line has become a tool for mathematical thinking.

Limitations

Employing the use of context and using models didactically were important pieces of the puzzle that early math reformers basing their practice on constructivism had not thought much about. While we planned problems as the starting point of instruction and we engaged children in rich discussions, we had not thought carefully about scaffolding and supporting development over time. The Dutch had. On the other hand, the Dutch had a model of learning that was linear. They spoke of learning lines and planned sequences that were crafted very tightly to move the whole class along this "line." They characterized learning as the development of progressive schematization. The problem with this approach to teaching is that learning is not linear. Mathematical thinking is comprised of a landscape of big ideas (structures), strategies (schemes), and mathematical models (Fosnot and Dolk, 2001, 2002). It requires the interpreting and explanation of part/whole relations and transformations (structures), the refinement of strategies and procedures (schemes), and the emergence of ways to mathematically model problems. The Dutch seemed to have an important *didactic* employing context and models; whereas, in the United States we had developed a reformed *pedagogy* turning classrooms into workshops and facilitating communities of discourse to facilitate development. In our collaborative project in New York, Mathematics in the City, we tried to combine the strengths of each.

MATHEMATICS IN THE CITY

Mathematics in the City began in 1995 funded by the National Science Foundation—a collaboration between the Freudenthal Institute and the City College of New York. It is primarily a professional development (inservice) project in mathematics education, although as we worked in classrooms with elementary teachers helping them reform their practice we researched several sequences employing context for the development of number and operation K-6; we developed mini-lessons using the open number line, open arrays, and double number line; and, we developed what we came to call

landscapes of learning. Rather than perceiving of scope and sequence as a learning line of progressive schematization, we perceived of learning as a non-linear journey comprised of many paths towards an horizon. The journey incorporates landmarks of big ideas, strategies, and models. Those interested in more detailed descriptions are referred to Fosnot and Dolk (2001a, 2001b, and 2002).

Assumptions Regarding Mathematics

Rather than a discipline or body of knowledge (concepts, skills) to be transmitted, we defined mathematics along the lines of Freudenthal's (1991) notion of "mathematizing"—the activity of interpreting, organizing, and constructing meaning of situations with mathematical modeling. But we expanded it to include mathematizing as a way of making meaning in the world. To us, it is a constructive process, which includes noticing patterns in special cases, analyzing why they are occurring, expressing them in some form of generality, and searching for elegance in the creation of strategy or proof. This "mathematizing" involves the setting up of quantifiable and spatial relationships, the constructing of patterns and transformations, the proving of them as generalizations and models, and the searching for elegance of solution. It involves the interpreting of one's "lived" world, mathematically. Mathematicians *create* mathematics to solve real life problems or to explain or prove interesting patterns, relationships, or puzzles in mathematics, itself—in a sense, they even mathematize their mathematical activity (Gravemeijer, 1997). They define problems and raise questions.

Constructing mathematical ideas is not only a cognitive activity; it is also a social one. Being a mathematician means being mathematical within a mathematical community. Mathematicians prove their ideas to each other. Ideas hold as accepted truths only in so far as the mathematical community accepts the proofs.

Assumptions Regarding Teaching

If mathematics is defined as "mathematizing," then what implications does this have for teaching? Learners need to be given the opportunity to search for patterns, raise questions, and construct ideas, strategies, and ways to mathematically model their "lived" worlds; and, they need to be invited to

defend their thinking to others in the community. The classroom, in a sense, becomes a *workshop* as learners investigate together. It becomes a mini-society—a community of learners engaged in mathematical activity, discourse, and reflection. Learners must be given the opportunity to act as mathematicians by allowing, supporting, and challenging their “mathematizing” of particular situations. The community provides an environment in which individual mathematical ideas can be expressed and tested against others’ ideas. Learners share perceptions with each other and with the teacher, and their ideas become modified, selected or deselected, as common (taken-as-shared, Cobb, 1996) meanings develop. This enables learners to become clearer and more confident about what they know and understand.

Frameworks for Learning and Teaching

Linear Frameworks. Only focusing on the structure of mathematics will lead to a more traditional way of teaching, one where the teacher pushes the children towards procedures or mathematical concepts. Historically curriculum designers employed such a framework. They analyzed the structure of mathematics and delineated objectives and goals along a line. The small ideas and skills were assumed to accumulate eventually into concepts (Bloom et. al., 1967). Many teachers held similar frameworks of learning and teaching. Each lesson, each day, was geared to a different objective, a different “it.” All children were expected to understand the same “it,” in the same way, at the end of the lesson. They were assumed to move along the same path; if there were individual differences it was just that some children moved along the path more slowly—hence, it was thought that some needed more time, or remediation.

Learning Trajectories. As educational reform based on constructivism has taken hold, curriculum designers and educators have tried to develop other frameworks. Most of these approaches are based on a better understanding of the learning of children and of the development of tasks that will challenge them.

Simon (1995), for example, describes a learning/teaching framework that he refers to as a “hypothetical learning trajectory”—hypothetical, because until the students are really working on a problem, we can never be sure what they will do, nor if and how they will construct new interpretations, ideas, and strategies. The teacher, however, expects the children to solve a problem

in certain ways; in fact, expectations are different for different children. Simon uses the metaphor of a sailboat.

You may initially plan the whole journey or only part of it. You set out sailing according to your plan. However, you must constantly adjust because of the conditions that you encounter. You continue to acquire knowledge about sailing, about the current conditions, and about the areas that you wish to visit. You change your plans with respect to the order of your destinations. You modify the length and nature of your visits as a result of interactions with people along the way. You add destinations that prior to the trip were unknown to you. The path that you travel is your [actual] trajectory. The path that you anticipate at any point is your ‘hypothetical trajectory’. (pp. 136-137)

As Simon makes clear, teaching is a planned activity. Teachers do not walk into their classrooms wondering what to do. They have a lesson planned and they anticipate what the children will do. With responses from children, teachers acknowledge the differences in thinking and in children’s strategies and they adjust their course in relation to these interactions.

Landscapes of Learning: Steps, Shifts, and Mental Maps

In our original work with teachers we too used the terminology of a hypothetical learning trajectory—even though we described it in terms of knowledge of models, strategies, and big ideas. But this terminology now seems too linear (Fosnot and Dolk, 2001). Learning—real learning—is messy business. We prefer instead the metaphor of a landscape (Fosnot and Dolk, 2001a, 2001b, 2002).

Learning and teaching are more a journey across a landscape than a trajectory, or learning line. The big ideas, strategies, and models—the structuring, the schematizing, and the modeling—serve as important *landmarks* for teachers to use as they plan, and as they journey with their children. As teachers design contexts for children to explore, the goal is to enable children to mathematize—to act on, and within, the situations mathematically using the landmark strategies. These progressive schematizations are the *steps* in the journey. The environment, the context, also is designed to facilitate discussion around big ideas because these

landmark ideas are major shifts in perspective—major *shifts* in structuring. As children model and represent their strategies, and as they develop generalized mental models of the part/whole relations for situations and operations, they construct *mental maps* that can eventually become tools to think with. For children, the landmarks epitomize their struggles in their journey to “make sense of” the world—they characterize the steps, shifts, and mental maps in the development of mathematizing. Teachers have horizons in mind when they plan—horizons like place value, or addition and subtraction. These appear from a distance and the journey thus seems to be a perpendicular line, a goal to be reached (a hypothetical trajectory), but in reality the journey is made up of many landmarks, and there are many ways to get to the horizon. The paths to these landmarks and horizons are not necessarily linear. Nor is there only one. As in a real landscape, the paths twist and turn; they cross each other, and are often not direct. Children do not construct each of these ideas in an ordered sequence. They go off in many directions as they explore, struggle to understand, and make sense of their world mathematically. Strategies do not necessarily affect the development of big ideas, nor the reciprocal. Often a big idea, like unitizing, will affect counting strategies; but just as often “trying out” new counting strategies that they have seen others use (like skip counting) will effect the development of unitizing. Ultimately, what is important is how children function in a mathematical environment (Cobb, 1997)—how they mathematize.

The Role of the Teacher

Viewing learning and teaching as a journey across a landscape makes teaching very difficult. The investigations must be rich and provide opportunities for many levels of mathematizing. Teachers must have an in-depth knowledge of the landscape in order to facilitate the journey. This means they must understand mathematics to be the *activity of mathematizing*. And, they must also understand the *development* of the landmarks—they must understand that learning is the result of autopoietic structures—of assimilation and accommodation. Often Piaget’s notion of assimilation has been misunderstood—assimilated, itself, with behaviorist beliefs. It has been described as a “taking in” of new information as long as learners are developmentally ready to understand it. From this perspective, learning is nothing more than the association of new concepts with prior concepts, and the cognitive reordering to build this connection. In contrast,

Piaget describes assimilation as the “acting on” a situation with initial organizing schemes—to make the situation “similar” to the present cognitive structures of the learner. This gets to the heart of constructivism. We know the world *through* the schemes and structures we use to explore it. Perturbations to these assimilatory schemes cause cognitive reordering (accommodation). These perturbations can come about when learners reflect on their actions and infer them to be insufficient or inefficient. They can come about when a cognitive structure is perturbed; or when two ideas seem to be contradictory. Thus, teachers need to become facilitators, provocateurs, and questioners. They must turn classrooms into workshops and structure discussions around big ideas and efficient strategies. And of course, pedagogical strategies need to be aligned with the process of learning, rather than with transmission.

Teacher Development

When learning and teaching are viewed from the perspective of a landscape, new models of pre-service and in-service education must be formulated. As we move into the 21st century, advances in digital technology are providing new promising possibilities for teacher education. Real teaching is based on visions of past practice, of strategies based on beliefs about learning, about epistemology, and the role of the teacher. Most teacher decision-making is split-second decision-making in the context of the teaching/learning act and is directly connected to the context of the classroom. Digital technology allows the teacher educator to immerse teachers in the context of real classrooms. Just as context is critical in learning to mathematize one’s “lived” world for students, the real world context of the classroom is critical to teachers as they learn to teach.

Digital production projects have traditionally used the technology to accompany texts for illustrating examples such as lessons, coaching models, and/or interviews (Carpenter, Franke, and Levi, 2003; West and Staub, 2003), or in other cases for modified lesson study (www.Lessonlab.com). Of exception are a few projects characterized by a more interactive, inquiry-driven approach to learning, e.g. MILE (Dolk et. al., 1996), and the University of Michigan's Space for Teaching and Learning Exploration (SLATE).

Current digital lab environments go even beyond those uses (see Dolk,

Fosnot, Cameron, and Hersch, 2004). They are interactive multi-level learning environments for the professional continuum. They bring the context of the classroom to the fingertips of the learner. Users are immersed in the study of children over time in exemplary classrooms; they examine the teacher's didactical employment of context; they can inquire about and analyze the pedagogy; they clip and paste moments from footage and build "landscapes of learning" (trajectories) that show children constructing "big ideas," developing strategies, and/or using mathematical models as tools; they are asked to solve mathematical problems in several ways and anticipate student strategies, which they subsequently examine; they design investigations and mini-lessons for the next day; they examine how the teacher in the environment continues and they subsequently analyze children's work to assess the effectiveness. They can even add clipped footage as hypertext-evidence to support arguments and provide examples in term papers and literature reviews, or as sample evidence of the NCTM standards. The materials are cross-platform, thus enabling users to work at home on assignments as well as in college classrooms, or on the internet. Inservice and preservice activities are interspersed to deepen teachers' abilities to analyze the mathematics in children's work, to explore the role of context in teaching mathematics, and to investigate teachers' questioning and choice of problems. Such materials provide a powerful context, an environment, for discourse and learning. [At this point in the speech I demonstrated one of the digital environments. These are now available through Heinemann Press].

CONCLUSION

Constructivism is a theory of learning, not a theory about teaching. But when one analyzes the theory, one can begin to formulate a reformed practice that supports rigor, empowerment, and the construction of genuine understanding. Over the last twenty years educators have been hard at work to reform the process of schooling accordingly—to bring our educational institutions into the 21st century. It is my hope that I have done justice in my work to the telling of the many attempts of reform, thereby contributing in some small fashion to the promises and possibilities of tomorrow.

*Since Cathy's keynote address at ACT 2002 in Houston, substantial portions of her speech have been published in her recently published book and are reprinted here by permission of the publisher from Fosnot, C.T. (2005). Epilogue Chapter in C.T. Fosnot (Ed.) Constructivism: Theory, Perspectives, and Practice, Second Edition.(New York:

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