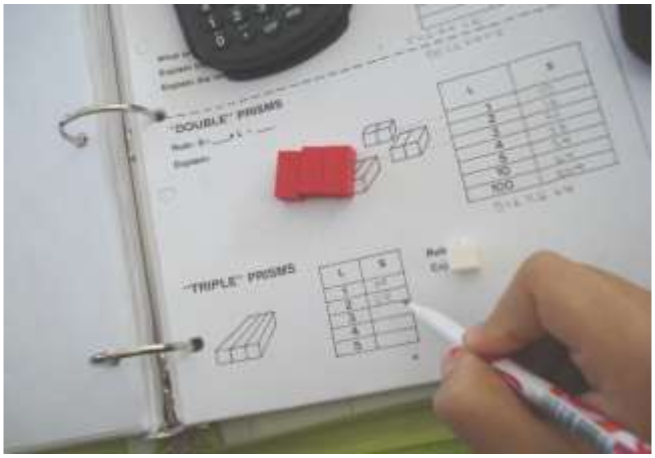
Introduction to the Algebraic Concepts of the Variable, the Pronumeral, and Equations by Means of the Constructivist Method



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School Year: 2008-2009

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(Video 1 - Algebra)
What is algebra, and what do people use it for? Do you remember the binomial of a rectangle? Have you ever used algebra outside of a school setting?

I believe that several among us could probably identify with some of the people interviewed in this video, which should make us reflect on ourselves as teachers, on whether we should be teaching the concepts that we are teaching, and in what way we are doing this. I have asked myself these questions countless times and have arrived at the conclusion that, in spite of the fact that the majority of people believe that they don't use algebra in their lives, they *do* in fact use algebra without noticing it; when they identify patterns, make generalizations, relate variables and make conclusions. The truth of the matter, I believe, is that this area of mathematics is one of the most useful to guide the thought process and resolve problems. However, as learning facilitators, we are doing something wrong regarding the utility and applicability of this learning, neither of which are clear or evident to the student. For this reason, it is not surprising that many students don't feel motivated or interested in this body of knowledge.

What are we doing wrong? Despite the many advances in the science of education and the most recent discoveries about *how* we learn, for many years the teaching of mathematics has remained static. If we compare an algebra textbook from 30 years ago with one printed this year, we find more aesthetic changes than changes in the learning experience. In the same way, in the majority of classrooms math teachers limit themselves to solving operations and problems on the chalkboard, waiting for their students to understand the algorithm and someday be able to transfer this to other contexts. This practice is typically carried out starting from a generalization, and proceeding to exemplify the generalization in particular cases.

Example

Find the value of y for $x=2$

$$y=6x + 7$$
$$y = 6(2) + 7$$
$$y = 19$$

What I am addressing is the theme: here, equations of the first degree. The teacher instructs the student to substitute and to realize the operations and necessary values in order to find the value of *y*. After this process, many students are capable of solving or simplifying a mathematical equation, but very few are capable of explaining what this can or does in fact represent.

The goal of this article is to share the analysis of several real-life experiences from the classrooms of the Nezaldi Institute, to introduce various arithmetical and algebraic concepts at the high school level by means of the constructivist method (described below), in hopes that these can be useful to answer certain questions addressed here, and to generate further inquiries.

ELEMENTS OF THE CONSTRUCTIVIST METHOD

1. To develop the affective or emotional component.
2. To use concrete material as well as persons as developing agents of thought.
3. To build on previous knowledge.
4. To design experiences and atmospheres for the learning of mathematics.
5. To make the process easier.

What effect does the emotional component have in the study of mathematics? How do we develop this?

One of the most important factors for successfully developing abilities in mathematics and science is self-efficacy. It is important that students find interest, enjoyment, and application and that they feel capable of doing the material in order to develop their potential. It is necessary that they fall in love with math, and with themselves solving math problems.

"I'm good at it!" Adolfo (Sophomore)
"I'm loving math" Mathias (Sophomore)

According to Carey and Gelman (1991), "there exist privileged understandings, and these are based in defined categories, notably physical, biological, causal and numerical concepts." However, if one is born with a natural interest to explore and learn mathematics, then why don't we hear more comments like those listed above? Why does one have to *rediscover* mathematics? Why, when it is so easy for a baby to recover their attention span in an activity after having lost it simply by varying the number of similar objects with which we are working, thus demonstrating the child's early interest for the concept of quantity, why it is so much work for an adolescent to become interested in algebra? Where did it lose its appeal? Why is it lost?

Students enter high school with a very definite idea about what the learning of mathematics means, and this has a strong impact on how they situate themselves in situations that require effort and understanding. Year after year, I have heard the following sentences spoken at the beginning of math courses: "I'm not good at math.", "I'm no good at fractions", "I'm not smart because I can't do math", and I have even heard the students' mothers comment "Help my son because he's no good at math, and he gets that from me."

And so the first thing to accomplish is to show these people the opposite is true and to present them with challenges appropriate to their level of difficult, utility and the impact they have on themselves and on others in order to improve their self-efficacy.

The affective component is also influenced when we are teaching and learning more complex mathematic concepts, what the experts call mathematics in transition. These are what cause recurring problems in students: fractions, decimals, percentages, negative numbers, and others. For example, fraction numbers don't behave like whole numbers, that is, their behavior does not accommodate easily to the "mathematical logic" that the student has been developing since their first years of life.

Example:

½ X ½ =1/4

"I must be doing something wrong because I multiply and the solution gets smaller." Matthew

Another example:

In an exercise of mental calculus the teacher asks the students to add ten plus three-fourths plus one-fourth and write the solution numerically.

"But how can I add ten as a number *and* as a fraction?"- Daniel

This demonstrates that the students had been working on fractions for four years and had arrived to a certain level of mechanized thinking; however, they had not internalized the concept of the fraction and its relationship with whole numbers.

Concepts like these cause confusion and adjustment can require a great deal of effort and understanding. Gardner (1983) indicates that in the search for efficient methods to increase the student's understanding, "educators should recognize the difficulties that students have in order to achieve a real understanding of certain themes and important concepts". We should assure ourselves that they expect these moments of confusion as part of the learning process and that it is through these challenging moments that they will acquire a greater understanding of the concepts. This has an impact on their level of self-esteem, shifting the emotional weight towards the area of knowledge.

Developing the Affective Component through showing the student that they can do it

The Theory of Even and Odd Numbers

The students were challenged to test the theory that all numbers, by means of a chain of operations, come out to 1 if we perform the following steps:

If the number is even you divide it by 2
If the number is odd you multiply it by 3 and add 1.

Geo demonstrated the theory of evens and odds in class using the most complicated challenge, which involves the number 27. She approached the teacher with a look of satisfaction, showing her work and said "Fern and I did it, more than 100 operations and we still got it."

This experience, by means of the application of simple operations of basic arithmetic, permitted the students to work as a team, plan, and prove a theory; these are all abilities and activities quite natural in mathematics. But the most significant thing was that the students surprised themselves when they could do it.

The Theory of Palindrome Numbers

Diego had resisted demonstrating the theory of palindromes which he claimed to know and had done before. When asked how he knew that the theory worked, he responded: "Just because." We discussed the importance of demonstrating the theory instead of settling for "Just because." During the exercise he came across various examples in which it wasn't as easy to arrive at the solution, making us doubt the validity of the theory. Once he finally solved the problem, he said "Now you see, I knew that all of them would come out as palindromes.", to which I responded "Now *you* see, I knew that you would be able to do it, but I had to demonstrate it, and you had to do both; be able to do it, and to *show* that you can do it."

Rules of Algebra- The Number-Transforming Machine

The introduction to rules of algebra was accomplished by means of a game developed by the Australian government. Students were challenged to discover what happened to a number when it was introduced into a number-transforming machine that changed numbers in the following way.

NUMBERS ENTERED	NUMBERS THAT CAME OUT
□	△
1	13
7	49

This idea of the machine is used because it is probable that students can relate this to a previously established concept of a machine that transforms an object into something different, in which a certain raw material enters and a distinct product comes out. Had this not been the case, the students would have had difficulty establishing relationships at this point between previously acquired concepts and the rules of algebra or equations I was introducing. This differs from other methods of teaching in that *here* you attempt to start from a concrete example to later arrive, by means of a series of life experiences, to a generalization about the meanings of the equation, the variable, and the pronumeral.

Matías: What's this about the triangle and the square?
Teacher: Look closely, instead of naming the numbers I put in and take out of the machine, I'm going to represent the ones I put in with a square, and the ones I take out with a triangle. That is to say, that when you see these symbols (the triangle and the square), you already know that I'm talking about the number that I put in and the number I take out of the machine.

At this point we aren't using algebraic language since the primary goal is to generate a real-life experience to which the student can relate a new mathematical concept and it's corresponding vocabulary. Just like a small child playing with different shapes before learning the names of those shapes, the goal here is that a student might play, explore, and experiment with equations before learning to call them variables, pronumerals and equations.

Teacher: What is happening with my numbers?
Andrés: They've been multiplied by 13.
Teacher: What do the rest of you think? What happens if I put in a number and multiply it by 13; what number will come out?
Javier: It gives you 13, but the number 7 you have to multiply by 7 so that 49 comes out.
Teacher: Aha! But if you change each number differently it would be like putting them each in different machines, and the numbers I am putting into the machine are all going into the same machine.

Here we are focusing the student's attention towards one of the fundamental principles of algebra which is it's trait of generalization, which is to say, the relationships that we establish between variables should be the same under that context we have established. This is the basic principal for the solution of problems.

Fernanda: Well this isn't possible.
Teacher: Hold on Fernanda, let's put a number into the machine and see what happens.
Fernanda: I don't know.
Teacher: Whatever number you want.
Jessica: 5 then.
Teacher: Alright, let's see what happens… We put in 5, and 37 comes out.

Those that already understand algebra and see this exercise might ask themselves; Why not introduce ordered and consecutive numbers that might facilitate detecting the rule of algebra more quickly? However, what this introduction aspires to accomplish is that the students might seek a relationship between the number that goes in and the number that comes out, not between the change that might occur between two entries. The goal is not that the student arrives at the correct mathematical answer, but to generate curiosity about *how* to solve the problem. In the same way, the final goal is for the student to find the problem challenging and, through the process, live out a satisfying experience, proving themselves capable of solving a problem that, at the beginning, they had considered to be very difficult.

Why is the use of concrete materials important at the High School level? What is the function of concrete materials as a developing agent of mathematic thinking?

Studies demonstrate that, in order to achieve any significant learning, the student must build new knowledge on top of previous experience. Algebra is the simplest form of generalized arithmetic; however, as was seen in the last dialogue, mathematical generalization is a concept that is practically new for students in the first year of high school, for which reason it is of primary importance that diverse real-life experiences be generated, utilizing different recourses such as concrete material, resources that can serve as a basis and support for the acquisition of new concepts and, later on, as a guide for representational work and abstraction.

I'm going to show a video in which we see Alan, a student in his first year at high school, working out operations with negative numbers. Alan comes from a traditional, bilingual immersion grade-school. He showed little or no intrinsic motivation concerning academic questions. He was accustomed to a basic scholastic discipline based on reward and punishment and was used to placing his self-efficacy on the opinions of third-parties. In math, Alan didn't feel capable of accomplishing complex challenges and knowledge gaps in basic operations of arithmetic often obstructed his progress.

(Video 2 - Alan doing operations with negative numbers)

In the video we see Alan enjoy working at mathematics. The concrete material permits us to detect some of the concepts that Alan has come to master, and others that are still raising difficulties for him. In the first problems of addition, from 12+-7, 13+-3 and -6 + 5, Alan models the operation correctly, getting pairs of zeros and arriving at the correct answer. However, he doesn't model the difference between the negative numbers 10 -22 and 11 - -8 correctly. The same thing happens with Geo and Adri in the following video.

(Video 3 - Adri and Geo)

Unlike Alan, Geo and Adri are students that were able to get the correct answers without the concrete material; however, we can see that having memorized the rule of signs does not in fact help them model their operations when these very operations imply subtraction in some places and addition in others. The opportunity to face these types of situations and to face their own errors will bring about a greater understanding of the concepts.

On occasion one might ask about the use of concrete material as a support, since some believe that the student could possibly develop a dependency on them. However, this could happen not because of the concrete material in question, but due to the way in which it is used. Bransford, Brown and Cocking (1999) suggest that what "a child can realize today with some assistance, he will be able to realize tomorrow independently."

Why is it important to build on previous knowledge?

I'm going to show you another example that also took place while we were practicing operations with negative numbers. Georgina, a high school freshman, was able to get the answers when adding negative numbers; however, she refused to use the algebraic scales to model her operations.

Georgina*-No, I don't understand it this way.*
Teacher*- What is it that you don't understand about the scales? Is it how they work?*

The algebraic scales (for those that haven't used them) represent equations and they have two trays on either side, one red tray for negative numbers or variables, and a yellow tray for positive numbers.

Georgina*- It's just that in my other school, they taught the rule of signs and I'm just supposed to take out the greatest number and then divide the* *numbers. And with the scale I just get confused.*

The preceding dialogue confirms what Bransford et al. (1999) explain, that "the transfer is affected by the grade in which people learn, understanding and not memorizing a series of facts or following a series of procedures."

"*It is easy to do algebra without a ruler, without understanding it.*" says Fernanda.

Why is it important to generate experiences and learning environments? How do we generate them? Why do we start from a real-life situation?

Starting from a real-life case or situation, we give the student the opportunity to discover algebraic rules for themselves, rules that describe an original situation and relate mathematics to daily life. The goal is that the student learn not only a body of knowledge but "when, where and why to use the knowledge that they are learning" (Whitehead in Bransford et al., 1999). Cognitive scientists "call the knowledge of experts 'conditioned' – this includes the specification of the context in which it is useful" (Bransford et al., 1999). The work of math in real situations permits the student to condition this knowledge and facilitate the transfer of this knowledge to other contexts.

Generating experiences for the Introduction of Descriptive Algebra- Painting Prisms

In the following experience algebra was used to describe a real-life situation, a situation that was used to continue reinforcing the concepts of the variable, pronumerals and the equation. Each student was given a set of small strips of paper, all of which varied in length from 1 to 10, and the students were asked to calculate the number of units squared (the measure of the area of one face of the blank cubic strip of paper) they would have to color in, were they to color each strip individually.

Adolfo: What do we have to do? We have to color them, but let's put them on the table graph.
Teacher: They ask you to calculate the number of units squared that you have to color if they ask you to paint the paper strips.
Fernanda: But what is a unit squared? I don't understand.
Teacher: Take the blank strip, the one of length "1". Look at the faces of the cube; that is a unit squared.
Fernanda: (*Indicating the* face) Is this the face?
Teacher: Yes, the unit face of the blank strip. How many squared units do we color in every strip?

Matías: Do we just count them up?
***Teacher*:** Count them up and then fill in the graph.
Jessica: I don't know how to fill it.
***Teacher*:** Look, take your blank strip. How long is it?
Jessica: Six.
***Teacher*:** It has six faces. But how tall is it?
Jessica: One.
***Teacher*:** On the graph, this column indicates the height of the strips and in this other column you write down the number of units squared that it has. What would you write down in the number of units squared for the strip that measures 1 in height?
Jessica:(*Indicating the face of the strip.)* 1,2,3,4,5,6.
***Teacher*:** And so, now do the same with the rest of the strips.
Jessica: With all 10?
Javier: You don't have to do all of them because you'll figure out the rule.
Jessica: I prefer to count them.

Fernanda: Like this?
***Teacher*:** Very good, Fernanda. What are you going to do now with that strip whose length is 100?
Fernanda: I leave it blank because I don't have a strip of 100.
Matías: No Fernanda, figure out the rule.
***Teacher*:** How do we find the rule?

Matías: Ok, you see how it keeps increasing. It's like the other exercise we did.
***Teacher*:** Examining and counting the strips you found the pattern Fernanda; now see *how* the number of faces will keep growing if we go on varying the height of the strips one by one.
Fernanda: Oh, like from 6 to 10, 4, and so forth, right?
***Teacher*:** Exactly. And from 10 to 14?
Fernanda: Four.
***Teacher*:** José, what does that four tell us?
José: That we're multiplying?
***Teacher*:** Exactly, the variable *L* represents the height that you multiply by four and produces that incremental gain. And what else do we have to do to get the rule that describes this pattern?
Javier: For example 4 X L plus 2 is 6. You add 2. (The teacher writes the formula on the blackboard)

S = 4 X L + 2

***Teacher*:** Now I want you to tell me what each one of the letters and numbers on our rule of algebra represent.
Daniel: The faces?
***Teacher*:** Show me which faces they are with the strips of paper.
Daniel: (Taking the blank unitary strip and counting the faces) One, two, three, four, Five? Six?
***Teacher*:** What happened when it got to four, Daniel?
Daniel: I counted 6.
***Teacher*:** Then our four doesn't represent all the faces of the strip. We said that you multiplied because it represented the increase. Which increase? (*Nobody answers*).

Take your red strip whose height is 3 and a green one whose height is 4. How many more faces does the green one have than the red one?

Matías, Adolfo, Javier and José find it quickly and mention that it is 4. For the rest of the group it was necessary to guide them and indicate which four units squared you increased on the strip. Everyone corroborated that the increase was the same with different pairs of strips.

Teacher: Show me what the number 2 represents on the ruler.
Javier: Two faces.
***Teacher*:** Which two? (*Nobody answers*). Which two faces are always the same? Look at the red strip whose height is 2, the green strip whose height is 3, and the purple one whose height is 4. Which are the faces that are always the same even thought the height of the strip may vary?
Adolfo: The top ones.
***Teacher*:** Exactly, the tops of the strips.

The experience continued and the students deduced the algebraic rule to color all of the rectangular prisms formed by two strips of paper. The concrete material permitted them to understand that each part of the algebraic rule represented something real, and that the operations used in the rule have a reason directly related to the situation. For the students this was very evident in the problem where the constant was +2 and in the case of the double prisms where the constant +4 referred to the units squared that are found on the extremities of the strips of paper and that didn't change as we increased the height of the strips and began to refer to the height of the strips by the variable "L", because it was what we were changing to fill up the table.

Facilitating the Process- The Number-Transforming Machine

In the example of the transforming machine given at the beginning of the presentation, the students were anxious to discover how to deduce the rule with which we had programmed our machine; however, the majority were unable to deduce the algebraic rule based only on the numbers entered and the numbers that came out.

⍙=□ X 3+1

⍙=□ X 2-1

⍙=□ X 4 + 3

⍙=□ X 5 - 1

⍙=□ X 3+ 4

ENTRIES	NUMBERS THAT CAME OUT
□	⍙
4	19
1	4

How do we facilitate the process? The students were given a series of algebraic rules and invited to discover which rule I had programmed into the machine that modified the numbers in the way shown in the table. The process was facilitated by having certain determined rules of *where* they could choose from. One of the students quickly found out which rule I had chosen and appeared very anxious to share his answer with the class. Two or three more students began to raise their hands little by little. Fernanda, who has certain difficulties in the area of mental calculation, began to get nervous because her classmates had figured it out quickly and she still struggled. I asked her to give me another number to enter and she chose 2. Her classmates told her that 2 would become 9. Fernanda began to try with the first rule with the help of her calculator, which gave her seven. We advised her to try to corroborate this with the same rule, what it would give her if she put in 4; she confirmed that it did not give her 19. We clarified for her that that couldn't be the rule then, and had to keep trying the rules out until we found the one that worked with all the entered numbers and their results. This brings us back to the importance of the character of generalization in mathematical rules.

Inverting the Process

After all the students had figured out which rule I had used, we inverted the procedure. I asked them to invent a rule to program into a different machine while I left the room for a moment. When I came back in, I asked them to give me the entered numbers and the numbers that came out on the following table.

ENTRIES	NUMBERS THAT CAME OUT
□	≙
3	18
4	23
5	28

I wrote the rule on the board: ≙ = □X 5+3. Their first reaction was one of surprise at having solved the equation so quickly, and this curiosity led them to look for where the 5 and the 3 came from on my algebraic rule. One of the students found that the numbers that came out in my equation increased by 5 and that was the reason that my rule had to multiply by 5; however, it wasn't clear why I had added 3 units. I asked them to multiply the 3 entries in the table only by 5 and to see if there was anything that caught their attention in those results. Upon doing these operations they realized that all of the results were 3 short from the number that came out of the machine. The students were excited to have found my trick out, and they wanted to find more rules from the table of values. In this practice negative numbers and fractions were included, which implied a more difficult challenge.

Final Reflections

The experiences commented here are, without a doubt, manifestations of a reality that, being a consequence of human nature itself, is often passed over imperceptibly as much in family circles (which is a shame), as in school. As teaching professionals we should always be alert: I refer to the immense capacity of deductive reasoning that students have, and their natural instinct to figure new things out, which, apart from teaching and learning, are quite evident in other things that the youth do outside of class.

The subject matter has an enormous level of transcendence in the education of adolescents, whose motivation and enthusiasm to overcome obstacles, even obstacles with a "high degree of difficulty", awakens at an early age. This is the general consensus concerning mathematics, thus they will not only learn this knowledge base, fundamental and basic for all other areas of knowledge, but that it might reside in their unconscious, like a lesson universally relevant and useful for the rest of their lives, that what was believed to be complicated is in fact easy and simple.

The fundamental meaning of these experiences in youth, besides having the self-esteem that one achieves by overcoming tests that were once thought complex and abstract—here we return to the stigmas of mathematics—resides not only in the stimulating sense of personal satisfaction and in the reactions of students when they discover themselves to be possessors of new knowledge, or when they discover their ability to transform a problem that was once dark, difficult, and boring, as with some frequency math is described in all of its forms, but instead this becomes a simple and clear exercise, and, besides that, interesting, fun and useful.

My reflections are not limited, however to the mere exercise of learning this discipline (although this might be considered sufficient to develop greater efforts in this direction). This achievement transcends the limits of learning math, to reach something more universal and, consequently, more important which is nothing less than "the joy of learning", which transforms the individual to the incessant search for knowledge, not as a chore or obligation but as a vital necessity of realization and source of pleasure, accomplishing a fundamental change in the life of a human being.

The necessity of "learning in order to learn" and of "lifelong learning", essential themes for any that might live the rest of their lives in the era of knowledge, is a necessity that cannot be satisfied without the achievement of this fundamental condition of acquiring "the joy of learning."

Centuries ago, the extraordinary French author J.P. Poquelin, known in the world of letters as Moliere, said: "What a beautiful thing it is to know something", and that, in the 21st century, is what we as teachers should hope to hear from our students, that they might feel that. That they might know and feel that knowledge, and in particular mathematics, that exact and sure science (as it has been described), is at the base of harmony: in music, in sculpture, in painting, in architecture, in the sciences, in poetry, in literature and in all the arts, and certainly, in the joy of knowledge itself, in emotions and in reason.

I have a challenge for teachers of our youth, to get them to say, paraphrasing Moliere: "What a beautiful thing it is to know mathematics", and the feeling you get when you achieve "the joy of knowledge", which does not impede the imagination, curiosity and creativity; this is our fundamental mission, to prepare these young people to live in freedom.

"The main thing I learned was, Algebra rules!", Daniel
"I learned Algebra this quarter.", Javier
"I'm going to do my next presentation in Spanish class about Algebra.", Adolfo

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