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MathNerds and Mathematical Knowledge for Teaching

Laurie O. Cavey
James Madison University, VA

W. Ted Mahavier
Lamar University, TX

G. Edgar Parker
James Madison University, VA

Alexander White
Texas State University

Introduction

Teaching mathematics in ways that empower students' mathematical curiosities and understandings requires, among other things, mathematical flexibility and insight on the part of the teacher (NCTM, 2000). Teachers know that flexibility is the name of the game when it comes to working within the school environment. For one, daily lessons are often interrupted due to factors such as school activities, adverse weather conditions or student behavior. Moreover, the ability to think flexibly about the content is also essential. For example, designing meaningful and challenging classroom tasks requires the teacher to think about which problems/tasks might be accessible to all students while, at the same time, providing opportunities for each student to develop important mathematical insights (Stein, Grover & Henningsen, 1996). Flexibility comes into play again as a teacher monitors the various approaches students take in thinking through a particular problem, which may lead to a number of different mathematical ideas and principles depending on how the teacher interacts with students' ideas (Ball & Bass, 2003). In fact, the ability to interpret a range of student ideas has been shown to be a critical factor in advancing young children's mathematical conceptions (e.g.; Hill, Rowan, & Ball, 2005; Yackel, 2002, and many others).

Many argue that this kind of flexibility in thought requires a teacher to call upon a mathematical knowledge base that is quite sophisticated (Conference Board of Mathematical Sciences, 2001). However, the nature of that knowledge base is not entirely clear and its complexity has become more evident over the last twenty years. Shulman's work (1986), which delineated subject specific knowledge for teaching into three categories: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge, was the catalyst of numerous studies aimed at understanding the development of mathematical content knowledge (Ball, 1990; Borko et al., 1992; Sherman, 1992; Simon, 1993) and pedagogical content knowledge (Ball, 1988; Blanton, Berenson, & Norwood, 2001; Eisenhart et al., 1993; Simon & Schifter, 1993). Later, Ma (1999) defined profound understanding of fundamental mathematics (PUFM) to incorporate basic ideas, multiple perspectives, connectedness, and longitudinal coherence, which illustrated connections between the mathematical and pedagogical knowledge required to teach children to understand mathematics. More recently, Ball & Bass (2003) and others have worked to develop a better understanding of the mathematical knowledge teachers need in their work. Classroom-based research has revealed evidence that teachers make use of mathematical knowledge that is distinct from that of mathematicians and others who are in a variety of mathematically intense professions (Bass, 2005; Cavey, Whitenack & Lovin, 2006; Hill, Rowan & Ball, 2005).

Following Hill, Rowan & Ball (2005) and Bass (2005), we use the phrase *mathematical knowledge for teaching* (MKT) to refer to the mathematical knowledge that teachers need in order to support the development of students' mathematical understandings. They described four categories of MKT to characterize the nature of mathematical knowledge needed for high quality teaching: *common mathematical knowledge* (e.g. concepts and procedures of adding whole numbers), *specialized mathematical knowledge* (e.g. analysis of alternative procedures for adding whole numbers), *knowledge of mathematics and students* (e.g. typical mistakes students make in adding whole numbers) and *knowledge of mathematics and teaching* (e.g. representations for adding whole numbers). Note that the category of *specialized mathematics knowledge* provides a way to distinguish between the mathematical knowledge that teachers use in their work from the mathematical knowledge mathematicians typically know and use.

MKT & Prospective Teachers

There is substantial evidence of prospective teachers who enter college with insufficient understanding of the mathematics they will teach and leave college having no opportunity to develop that understanding (Conference Board of Mathematical Sciences, 2001, p. 5). It is not necessarily an issue of prospective teachers needing more college mathematics courses, what one might consider the subject matter purists point of view. Rather, it is an issue of providing opportunities for prospective teachers to make connections between college mathematics and the mathematics they will teach and to begin understanding the subtleties involved in interpreting and responding to students' ideas.

As teacher educators, we recently explored new ways to develop aspects of our prospective teachers' MKT by engaging them in activities designed to apply their mathematical knowledge to secondary classroom situations. In particular, our prospective teachers interacted with local school district students through a web-based technology and assisted students with their mathematics work. In the past we used case studies, videotaped interviews with children, and samples of student work to engage prospective teachers in thinking about how to interpret and respond to middle and high school students' mathematical ideas and questions. While these techniques have been useful for bringing critical ideas to the forefront, such approaches remain purely hypothetical (and thus somewhat stale), as there is no real potential for the prospective teacher to interact with students. In an attempt to move beyond the hypothetical, but within a controlled and supervised environment, we used an online question-and-response service MathNerds (www.mathnerds.com) as a medium for mathematical conversations between prospective teachers and middle school students.

As teachers, we realize that during face-to-face interactions with students, a teacher may have little time (often, just a moment) to determine what a student is really asking and whether or not that student might be able to answer her own question. The online exchanges afford extra time for the prospective teachers to think about what the student might really be asking and to formulate responses that are both mathematically and pedagogically appropriate. It is important to note that one central goal in our work with prospective teachers is to help them understand the significance of attending to students' ideas as a critical part of teaching with a constructivist mindset. Like Bruner (1986), we emphasize process over product and like von Glasersfeld (1995), we acknowledge both the individual and collective nature of knowledge and its development. The online dialogues are an important part of our instruction that encourages prospective teachers to attend to individual student's thought processes and resulting strategies, while enabling us to track prospective teachers' resulting pedagogical and mathematical actions. In short, the online dialogues make it possible for us to model the pedagogical mindset we hope to instill in the prospective teachers.

In this paper, we share some of the insights gained from using online exchanges to engage prospective teachers in thinking deeply about how to interpret and respond to student questions. First, we share some background about the project and, in particular, the online environment that has made the project possible.

MathNerds

For more than ten years, the non-profit MathNerds (Dawkins, De Angelis, Mahavier, Stenger, 2002) has provided a free, web-based, question-and-response service supplying guidance (but not answers) in mathematics to students around the world. Over the past three years, the site has responded to approximately 1,500 questions per month with an average response time of approximately 16 hours. The team consists of more than 100 volunteers sharing a love of mathematics and a willingness to give time each week for nothing more than an occasional "Thank You" message. Most hold doctoral degrees in mathematics or mathematics education and all are tested initially and monitored. Volunteers represent a broad spectrum of society, including government employees, graduate students, high school teachers, industry employees, and faculty ranging from community colleges to research institutions. Through personal profiles, volunteers control the number of questions they receive and the categories (K-12 through graduate) in which they receive questions. Clients submit questions online that are routed randomly to the volunteers who have agreed to respond to questions in that category and who have not met their weekly quota. In alignment with the constructivist perspective, MathNerds has a strong commitment to inquiry-based education, teaching people to teach themselves and striving to avoid contributing to the abuse of the internet by doing homework, take-home tests, or school-related projects. Rather, volunteers are committed to providing individual guidance, references, and hints -- not answers per se.

In recent years, MathNerds has developed Mentoring Networks to connect school districts to local universities. During the fall of 2005, MathNerds entered into a partnership with Harrisonburg City Schools (HCS) in Virginia, James Madison University's College of Education, and James Madison University's Department of Mathematics and Statistics. Following MathNerds' inquiry-based question-and-response model, we developed and delivered a pilot program where middle and high school students submitted questions through the website that were routed directly to prospective teachers in one of the author's classes. These questions and responses were carefully monitored by each of the authors.

Preliminary analysis of the questions posed during the pilot program and reflections on our experiences have prompted us to question exactly how our prospective teacher's MKT might be activated through participation in the online dialogues. To illustrate the complex nature of our investigation, we highlight one question that was posed by a middle school student during the pilot program and consider the mathematical knowledge that a teacher might reflect upon to answer this question. In addition to providing two perspectives for thinking about how to respond to the question, we invite the reader to also reflect on how he/she might respond. By doing so, we attempt to illustrate some of the ways in which experienced teachers apply their mathematical knowledge to the task of supporting students' mathematical understandings.

Student Question and Possible Responses

The MathNerds environment requires clients to submit "work" along with each actual "question." So, typically, the volunteer has some evidence of the client's thought process that may inform conjectures about what the client might understand or what to ask the client for clarification. In addition, the information from the question and work informs the volunteer about significant mathematical ideas to which the client's question might lead given an opportunity to continue the dialogue. The ultimate challenge is often related to determining how much to "tell" so that the client might be encouraged to continue thinking about the mathematics and engaging in the dialogue. A question posed (along with "work done") by a middle school student during the 2005 pilot is provided below followed by two possible responses and discussion about the rationale for each response.

> Student Question**:** *How do you divide a fraction?*
> *5/10 divided by 3/5*
> Work done: *5/10 divided by 3/5 = .3?*

Before reading the two perspectives that we offer, we encourage the reader to take a few minutes to develop a possible response and rationale. Questions to consider while doing so include: What might the student understand about division by fractions? What evidence is there to support this conjecture? What questions for the student might reveal what the student is really asking? What are the important mathematical ideas related to this question?

We provide the following two perspectives to illustrate the kinds of mathematical ideas that teachers might reasonably think about while developing a response to the student's question. These two responses are not intended to illustrate what might be considered "best practice" or as the only possibilities for thinking about the student's question. Rather, they are presented to demonstrate the range of mathematical ideas that a mathematics teacher might consider while trying to engage students in thinking about mathematics in meaningful ways for the students. The authors developed the responses after each created his/her own initial response.

Perspective #1. One way to initially think about a response to the student's question is to decide whether to take an algebraic or geometric approach. From this perspective, the goal with the initial response might reasonably be to find out if the student understands the meaning of reciprocal. If so, one could choose to take an algebraic approach. If not, a geometric approach may be more accessible, psychologically speaking, to the student. With these ideas in mind, the following response was crafted.

> *Response #1. Hello. Let me first ask you a question. What do you mean when you write 5/10? Do you mean 5 units split into 10 equal size pieces or do you mean 5 multiplied times the number that, when multiplied by 10, gives the answer 1? Or do you mean something else? Thank you!*

Given the opportunity to take a geometric approach, a follow-up response could be crafted to focus the student's attention on building two rectangles from the same 'unit' rectangle (representing 1/10). The goal of this approach might be to help the student understand that 5/10 is one unit rectangle smaller than 3/5 (6/10) and that determining 5/10 divided by 3/5 is equivalent to determining how much of 3/5 fits into 5/10. In this case, only 5 of the 6 unit rectangles fit, so the answer is 5/6.

Perspective #2. One might wonder if the student mistakenly computed 5/10 * 3/5 = 3/10 = 0.3, and thus forgot the "flip" in "flip and multiply." Another possibility is that the student computed 5/10 * 5/3 = 5/6 and is asking how to divide 5 by 6. Note that if the student divided 5 by 6 using long division which resulted in 8/10 with remainder 2/6 then they might well have dropped the 8/10, simplified 2/6 to 1/3 and then decided 1/3 was equivalent to 0.3. Thus, there are at least two ways the student might have concluded the mistaken answer. Upon initial inspection, the former case seems more likely, but the use of the phrase "divide a fraction" makes one wonder if the student is actually asking about doing the division implied by the fraction. With these ideas in mind, the following response was composed.

> *Response #2. Dear Student, Thank you for submitting such an interesting question! When you write, "how do you divide a fraction" do you mean, how do we convert a fraction such as 5/8 into a decimal expansion such as 0.625? Or do you mean how do we divide a fraction by another fraction -- for example how do we divide 5/10 by 3/5?*
>
> *Assuming that you are asking how to divide the two fractions, let's see how we might check if your answer is correct. If I divide 12 by 4 and get 5 then I would multiply 5 by 4 to see if I get 12 back. Oops. I got 20, so 12/4 must not be 5!!! To check your answer, we need to multiply 0.3 (or 3/10) by 3/5 and see if we get 5/10. Try that and write back to let me know what you get! I'm not sure if you know about "reciprocals," but it might help to remember that dividing by a number is the same as multiplying by its reciprocal. The reciprocal of a/b is b/a.*
>
> *I am very interested in this problem, so if you could show me the STEPS that you took to get the 0.3 then I think I can get a better understanding of how you are solving the problem and be more helpful.*
> *Good luck and please write back.*

In this case, a follow-up response might provide an opportunity to emphasize the connection between multiplication and division, the meaning of reciprocal, or the algorithm for converting fractions to decimal form.

MKT in Action

Aspects of MKT are put into action when determining how one might effectively respond to a student's question. In fact, we argue that *specialized knowledge of mathematics*, *knowledge of mathematics and students, and knowledge of mathematics and teaching* are of particular importance. Certainly *common mathematical knowledge* is essential for understanding the basic concepts related to a given question. However, the other categories draw attention to the dimensions of MKT that enable the teacher to respond in ways that are informed by the practices of quality mathematics teaching and support the potential to extend the student's knowledge beyond the initial question, making connections to the big ideas of mathematics and various ways of interpreting mathematical ideas. In other words, we argue that having mathematical knowledge *beyond* common mathematical knowledge makes the capacity for formulating a response that has the potential to engage the student in careful, appropriate, and even sophisticated mathematical thought more likely.

Specialized Knowledge of Mathematics. When interpreting a student's question, a teacher may use specialized knowledge of mathematics while analyzing the student's work to identify both the significant mathematical ideas to which to attend in response (either initially or with subsequent responses) and/or to justify the mathematical accuracy of the student's work. In relation to the student question examined in this paper, the significant mathematical ideas include reciprocals and division by fractions (perspectives #1 & 2), the meaning of rational numbers (perspective #1) and the relationship between multiplication and division (perspective #2). Although there was minimal "work done" by this student, part of the thinking in perspective #2 involved analyzing different ways that the student may have come up with the answer of 0.3. This kind of thinking is a nice example of the distinctive kind of mathematical work done by teachers who are attempting to think through and validate the mathematical work of students.

Knowledge of Mathematics and Students. At the very least, knowledge of mathematics and students comes into play when interpreting the mathematical language used by students and when formulating a response to a student's question. Interpreting the language used by students involves understanding mistakes students tend to make both with terminology and mathematical procedures. As noted in perspective #2, a common mistake related to division by a fraction is to forget the “flip” in the “flip and multiply” procedure. In formulating a response, the teacher might consider how to respond in a way that will connect to what the student understands so that the student will want to continue working on the problem. This was illustrated in perspective #1 by suggesting two ways that the student may be interpreting the rational number 5/10 and was illustrated in perspective #2 by offering two interpretations of the student’s question. Both of these approaches make an attempt to connect with how the student is thinking by making suggestions, but ultimately require the student to decide independently.

Knowledge of Mathematics and Teaching. Knowledge of mathematics and teaching also plays a part in the process of formulating a response to a student’s question. In general, considerations may be given to how to communicate with words, symbols and/or pictures in a way that is consistent with the constructivist perspective. In our examples, perspective #1 attends to a psychological issue when considering whether or not an algebraic approach might be accessible to the student. By asking the student to choose between two interpretations of a reciprocal (or to offer another alternative), this perspective aims to set the stage for developing a dialogue to which the student can connect in a meaningful way. In addition, both responses invite the student to write back by asking a question and thank the student for asking the question. Such actions illustrate one primary goal of the constructivist perspective; that students be empowered to think for themselves (von Glasersfeld, 1995, p. 176).

MathNerds and Developing MKT

When prospective teachers engage in formulating responses to students’ questions, we noticed that it seems especially beneficial for them to see and hear about what other people are thinking regarding a particular student’s work. At the beginning of the semester, our prospective teachers tend to be very reluctant to submit responses without first checking with their methods instructor. Note that they have a rubric that is designed to serve as a guide to the process of responding, but this is often the first time they have been asked to think about how to respond to a question in a way that is direct (showing careful attention to the mathematics by drawing attention to the critical issues at hand) and encourages the student to take ownership of the work (not telling all). To help these prospective teachers feel more confident and be more competent, we have, at times, used one student’s question to generate a class discussion about how to respond, much like what we have done in this paper. These discussions provide an opportunity to emphasize the fact that, as teachers, we are always limited in what we can say about a student's thinking. These limits come from what we know about how students make sense of mathematics, the psychological perspective (i.e. constructivism) we rely upon and our own knowledge of mathematics, but more importantly, it is impossible to really know how another person is thinking. When it comes to analyzing and making sense of student thinking, there is only evidence and conjecture. These conversations also afford us opportunities to revisit the complexities involved in making sense of mathematics, the importance of clear communication of mathematical ideas and connections between topics across the curriculum. At other times, in addition to presenting a student's question and work, we have also presented a given response as a point of discussion. By doing so, we hypothesize about responses that may solicit more information about student thinking.

The potential gains in MKT are first and foremost dependent on the student's question. It is the student's question that determines the kind of mathematical knowledge required to respond. Some of our prospective teachers have received great questions to think about and to which to respond, whereas others have received information-type questions that have not afforded opportunities to think deeply about students' ideas. In addition, the potential for growth in MKT may also be dependent upon the number of different perspectives that one is able to consider. For a novice (or an expert), having the advantage of hearing about other ways of thinking about how to respond, and the mathematical ideas influencing that response, is a sure way to expand one's thinking even if it does not happen to change one's mind about a preferred way to respond.

Given these considerations, we are currently experimenting with other ways to incorporate online mathematical dialogues via MathNerds in our mathematics methods courses. One approach that seems particularly promising is to make it possible for a group of prospective teachers to work together to formulate a response to a student's question. The programming necessary to support this idea is currently underway. In the meantime, as particularly promising questions come up, we are taking time to consider them as a class to generate conjectures about student thinking and responses that might solicit more information about students' mathematical ideas. In addition, we are in the process of identifying student questions that have successfully generated meaningful discourse about the student's mathematical thinking and the MKT required to effectively respond to that student's question. Ultimately, we hope to establish a framework in support of particular MKT lessons, some of which will bring specialized mathematical knowledge considerations to the forefront. Others will make it possible to consider more of the pedagogical side of formulating a response.

Final Remarks

As teacher educators, we aim to provide experiences that enable prospective teachers to recognize the complex nature of interpreting students' mathematical work as well as the significance of engaging in doing so as a means of empowering their own students' ability to think and reason mathematically. In the process, we strive to provide opportunities for prospective teachers to wrestle with the intricacies of making sense of school mathematics and to model pedagogical approaches consistent with the constructivist perspective.

Opportunities to reflect on the mathematics that prospective teachers will be teaching, including communication, notation, connections, meaning, their own conceptions, and common mistakes made by students, has been shown to be a critical component of teacher preparation (Conference Board of Mathematical Sciences, 2001). In an attempt to create these kinds of experiences with real students, it becomes difficult to assess potential gains and regulate the kinds of interactions and questions posed by students. On the other hand, the prospective teachers who participated in the mentoring network pilots were exposed to a broader perspective with respect to both the mathematical knowledge and the pedagogical approaches associated with answering actual questions posed by the clientele they will serve upon graduation. In addition, the relationships established between the mathematics educator and the mathematicians have furthered each partner's understanding of the others' work and encouraged the start of additional pilots[2] at Texas State University, Lamar University and surrounding school districts.

[2] For more information about the Mentoring Networks, see www.mathnerds.com/mathnerds/mentoringnetwork.

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